Discriminant of a generic projection of a minimal normal surface singularity

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Abstract

Let $(S,0)$ be a rational complex surface singularity with reduced fundamental cycle, also known as a *minimal* singularity. Using a fundamental result by M. Spivakovsky, we explain how to get a minimal resolution of the discriminant curve for a generic projection of $(S,0)$ onto $(\mathbb{C}^2,0)$ from the resolution of $(S,0)$.

The material in this Note is organized as follows : § 1 recalls the definitions of polar curves, discriminants and a remarkable property of transversality due to Briançon-Henry and Teissier (thm. 1.2). For minimal surface singularities, a theorem due to M. Spivakovsky describes the behavior of the generic polar curve (cf. § 2). We use this theorem in § 3 to prove two lemmas relating on the one side the resolution of the generic polar curve to the resolution of a minimal surface singularity, and on the other side, the polar curve and the discriminant. Gathering these results, we give our main theorem in § 4, which provides us with a combinatorial way to describe the discriminant.

1 Polar curves and discriminants

Let $(S,0)$ be a normal complex surface singularity $(S,0)$, embedded in $(\mathbb{C}^N,0)$: for any $(N-2)$-dimensional vector subspace D of $\mathbb{C}^N$, we consider a linear projection $\mathbb{C}^N \to \mathbb{C}^2$ with kernel D and denote by $p_D : (S,0) \to (\mathbb{C}^2,0)$, the restriction of this projection to $(S,0)$.

Restricting ourselves to the D such that p_D is finite, and considering a small representative S of the germ $(S,0)$, we define, as in [11] (2.2.2), the *polar curve* $C_1(D)$ of the germ $(S,0)$ relative to the direction D, as the closure in S of the critical locus of the restriction of p_D to $S \setminus \{0\}$. As explained in loc. cit., it makes sense to say that for an open dense subset of the Grassmann manifold $G(N-2,N)$ of $(N-2)$-planes in $\mathbb{C}^N$, the space curve $C_1(D)$ are *equisingular* in term of strong simultaneous resolutions.

Then we define the *discriminant* Δ_{p_D} as (the germ at 0 of) the reduced analytic curve of $(\mathbb{C}^2,0)$ image of $C_1(D)$ by the finite morphism p_D.

Again, one may show that, for a generic choice of D, the discriminants obtained are *equisingular germs of plane curves*, but we will need a much more precise result, that demands another definition (cf. [4] IV.3) :

Definition 1.1. Let $(X,0) \subset (\mathbb{C}^N,0)$ be a germ of reduced curve. Then a linear projection $p : \mathbb{C}^N \to \mathbb{C}^2$ will said to be *generic* w.r.t. $(X,0)$ if the kernel of p does not contain any limit of secants to X (cf. [4] for an explicit description of the cone $C_5(X,0)$ formed by the limits of secants to $(X,0)$).

We now state the following transversality result (proved for curves on surfaces of $\mathbb{C}^3$ in [5] thm. 3.12 and in general as the "lemme-clé" in [14] V (1.2.2)) :

Theorem 1.2. *Let $p_D : (S,0) \to (\mathbb{C}^2,0)$ be as above, and $C_1(D) \subset (S,0) \subset (\mathbb{C}^N,0)$ be the corresponding polar curve, then there is an open dense subset U of $G(N-2,N)$ such that for $D \in U$ the restriction of p to $C_1(D)$ is generic in the sense of def. 1.1.*

Definition 1.3. For all D in the open subset U of thm. 1.2, the discriminant Δ_{p_D} are equisingular in the sense of the well-known equisingularity theory for germs of plane curves (cf. e.g. the account at the beginning of [4]) : we will call this equisingularity class the *generic discriminant* of $(S,0)$.

2 Polar curves for minimal singularities of surface after Spivakovsky

We first recall how one may define a minimal singularity in the case of normal surfaces (cf. [13] II.2) :

Definition 2.1. A normal surface singularity $(S,0)$ is said to be *minimal* if it is rational with reduced fundamental cycle (see [1] for these latter notions).

Let $\pi : (X,E) \to (S,0)$ be the minimal resolution of the singularity $(S,0)$, where $E = \pi^{-1}(0)$ is the exceptional divisor, with components L_i. A *cycle* will be by definition a divisor with support on E i.e. a linear combination $\sum a_i L_i$ with $a_i \in \mathbb{Z}$ (or $a_i \in \mathbb{Q}$ for a $\mathbb{Q}$-cycle).

Considering the dual graph Γ associated to the exceptional divisor E (cf. [13] I. 1) in which each component L_x gives a vertex x and two vertices are connected if, and only if, the corresponding components intersect, the minimal singularities have the following easy characterization (cf. loc. cit. II. 2) :

Lemma 2.2. *Let Γ be as above the dual graph associated to the minimal resolution of a normal surface singularity $(S,0)$. For each vertex $x \in \Gamma$, one defines its weight $w(x) := -L_x^2$ (self-intersection of the corresponding component L_x) and its valence $\gamma(x)$ which is the number of vertices connected to x. Then $(S,0)$ is minimal if, and only if, Γ is a tree and for all $x \in \Gamma$, $w(x) \geqslant \gamma(x)$.*

To two vertices $x, y \in \Gamma$ (which is a tree), we associate the shortest chain in Γ connecting them, which we denote by $[x, y]$. The *distance* $d(x, y)$ is by definition the number of edges on $[x, y]$.

In [13] III. 5, generalizing an earlier work by G. Gonzalez-Sprinberg in [7], M. Spivakovsky further introduces the following number s_x associated to each vertex $x \in \Gamma$. If $Z.L_x < 0$ (where . denotes the intersection number) then put $s_x := 1$ (and x is said to be non-Tyurina). Otherwise x is said to be a Tyurina vertex, then denote Δ the Tyurina component of Γ containing x (i.e. the maximal connected subgraph of Γ containing only Tyurina vertices), and put $s_x := d(x, \Gamma \setminus \Delta) + 1$.

These numbers s_x coincide, in the special case of minimal singularities, with the *desingularization depths* introduced in [12] p. 8.

Let x, y be two adjacent vertices : the edge (x, y) in Γ is called a *central arc* if $s_x = s_y$. A vertex x is called a *central vertex* if there are at least two vertices y adjacent to x such that $s_y = s_x - 1$ (cf. loc. cit.).

Eventually, we define the following $\mathbb{Q}$-cycle Z_Ω on the minimal resolution X of $(S, 0)$ by :

$$Z_\Omega = \sum_{x \in \Gamma} s_x L_x - Z_K, \tag{1}$$

where Γ is the dual graph of the resolution, and Z_K is the numerically canonical $\mathbb{Q}$-cycle [1].

One may now quote the important theorem 5.4 in loc. cit. in the following way :[2]

Theorem 2.3. *Let $(S, 0)$ be a* minimal *normal surface singularity. There is a open dense subset U' of the open set U of thm. 1.2, such that for all $D \in U'$ the strict transform $C_1'(D)$ of $C_1(D)$ on X :*
a) is a multi-germ of smooth curves intersecting each component L_x of E transversally in exactly $-Z_\Omega.L_x$ points,
b) goes through the point of intersection of L_x and L_y if and only if $s_x = s_y$ (point corresponding to a central arc *of the graph). Further, the $C_1'(D)$, with $D \in U'$ do not share other common points (base points) and these base points are simple, i.e. the $C_1'(D)$ are separated when one blows-up these points once.*

3 From polar curves to discriminants, key lemmas

Lemma 3.1. *Let $(S, 0)$ be a* minimal *normal surface singularity, embedded in $\mathbb{C}^N$ and $\pi : X \to (S, 0)$ its minimal resolution. It is known that π is (the restriction to S of) a composition $\pi_1 \circ \cdots \circ \pi_r$ of point blow-ups in $\mathbb{C}^N$. We*

[1] uniquely defined by the condition that for all $x \in \Gamma$, $Z_K.L_x = -2 - L_x^2$ since the intersection product on E is negative-definite

[2] see also the account in [10] (7.4), just beware that one term is missing in the formula giving $m_x := -Z_\Omega.L_x$ there.

claim that this composition of blow-ups is also the minimal resolution of the generic polar curve $C_1(D)$ for $D \in U'$ as in thm. 2.3.

Proof. The fact that π is a composition of point blow-ups is general for rational surface singularities (for a non-cohomological proof in the case of minimal singularities, see [3] 5.9). Conclusion a) in thm. 2.3 certainly gives that π is a resolution of $C_1(D)$. We prove that this resolution is minimal : among the exceptional components in X obtained by the last point blow-up, there is a component L_x corresponding either to a central vertex of Γ or to the boundary of a central arc.

If L_x corresponds to a central vertex, one computes from (1) page 3, the number of branches of $C'_1(D)$ intersecting L_x, i.e. $-Z_\Omega.L_x = -(\sum s_y + (s_x + 1)L_x^2 + 2)$. By the definition of a central vertex (before thm. 2.3), this must be at least two, which proves that these branches are not separated before L_x is obtained.

If L_x is the boundary of a central arc, let L_y be the other boundary : then both L_x and L_y appear as exceptional components of the last blow-up $\pi_r : X \to S_{r-1}$ at 0_{r-1}. Now, the strict transform of $C_1(D)$ at the point 0_{r-1} can not be smooth. Indeed, by an argument in [8] 1.1, if it were smooth, then its strict transform $C'_1(D)$ on X, smooth surface, would go through a smooth point of the exceptional divisor. □

Lemma 3.2. *For $D \in U'$ as in thm. 2.3, the polar curve $C_1(D)$ on $(S, 0)$ has only smooth branches and branches of multiplicity two, the latter being exactly those whose strict transform go through a central arc as in b) of theorem 2.3.*

Proof. For $D \in U'$ one may compute the multiplicity $e(\Delta_{p_D}, 0)$ of the discriminant en 0 by the following (cf. e.g. [3] § 4.4 : here the divisorial discriminant is reduced by genericity) :

$$e(\Delta_{p_D}, 0) = \mu - 1 + e(S, 0), \tag{2}$$

where μ is the Milnor number of a generic hyperplane section of $(S, 0)$ and $e(S, 0)$ is the multiplicity of the surface. A well-known formula (see e.g. [2] prop. 5, taking $I = m$) for μ reduces, for $(S, 0)$ minimal, to $\mu = 1 + Z.Z_K$. In turn, $e(S, 0) = -Z^2$ (cf. [1]), which in (2) reads :

$$e(\Delta_{p_D}, 0) = Z.Z_K - Z^2. \tag{3}$$

On the other hand, the number n_b of branches of Δ_{p_D} is described by thm. 2.3 : denote by n_{bs} the number of those branches which go through a central arc and so are counted twice in $Z.Z_\Omega$, then :

$$n_b = -Z.Z_\Omega - n_{bs}. \tag{4}$$

Using expression (1) for Z_Ω in (4), and then using (3) yields : $n_b = -\sum_{x\in\Gamma} s_x L_x.Z + Z.Z_K - n_{bs}$ and $n_b = e(\Delta, 0) - n_{bs} - \sum_{x\in\Gamma}(s_x - 1)Z.L_x$, but, by definition $s_x = 1$ if $Z.L_x \neq 0$, whence :

$$n_b = e(\Delta, 0) - n_{bs}. \tag{5}$$

Since we know from the proof of lemma 3.1 that all the branches of $C_1(D)$ counted in n_{bs} are actually singular, (5) proves the whole assertion of the current lemma. $\square$

Corollary 3.3. *Take the chain of point blow-ups over $(\mathbb{C}^N, 0)$ that gives the minimal resolution of $(C_1(D), 0)$ for $D \in U'$. Then performing over $(\mathbb{C}^2, 0)$ the "same" succession of blow-ups (this makes sense because of footnote 3), we get the minimal resolution of the plane curve $\Delta_{p_D} = p_D(C_1(D))$.*

Proof. Since, by lem. 3.2, the multiplicity of the branches of $C_1(D)$ is at most two, these branches are plane curves and so are equisingular to their *generic* projection by p_D (here we use thm. 1.2) : so much for the branches. Further, by another result of Teissier's (see [14] Chap. I (6.2.1) and remark p. 354) a generic projection is bi-lipschitz, which implies that it preserves the contact between branches.[3] $\square$

4 Statement of the main result

Gathering the results from lemma 3.1 to cor. 3.3, we obtain :

Theorem 4.1. *Let $(S, 0)$ be a minimal normal surface singularity, embedded in $\mathbb{C}^N$ and $\pi : X \to (S, 0)$ its minimal resolution, which is a composition $\pi_1 \circ \cdots \circ \pi_r$ of point blow-ups in $\mathbb{C}^N$. Let $\Delta_{S,0}$ be the generic discriminant of $(S, 0)$ (cf. def. 1.3). Then, performing over $(\mathbb{C}^2, 0)$ the "same" succession of blow-ups (cf. footnote 3), we get the minimal resolution of the plane curve $\Delta_{S,0}$.*

We claim that this result, together with thm. 2.3 gives an easy way to get a combinatorial description of $(\Delta_{S,0}, 0)$:

Notation 4.2. i) We denote by Δ_{A_n} the generic discriminant of the A_n surface singularity, which is the equisingularity class of the plane curve defined by $x^2 + y^{n+1} = 0$.

ii) We denote by δ_n the generic discriminant of the singularity which is a cone over a rational normal curve of degree n in $\mathbb{P}^n_{\mathbb{C}}$: it is defined by $2n - 2$ distinct lines through the origin.

The assertion in ii) follows from the fact that $C_1(D)$ is the cone over the critical locus of the projection from the rational normal curve onto a line, which has degree $2n - 2$ by Hurwitz formula.

[3] Indeed, the contact between two branches $\gamma_1(t)$ and $\gamma_2(t)$ which are both of multiplicity one or two, that we define as the number of blow-ups to separate them, may be read from the order in t of the difference $\gamma_1(t) - \gamma_2(t)$, which is a bi-lipschitz invariant. Since we blow-up always in the "same chart" these blow-ups actually dominate the blow-ups of the plane, as claimed in the corollary.

We need to introduce several subsets of a dual graph Γ : we denote by $\Gamma_{NT} = \{x_1, \ldots, x_n\}$ the set of Non-Tyurina vertices in Γ, which are here the $x \in \Gamma$ such that $w(x) > \gamma(x)$ (notation as in lem. 2.2).

We denote by $\mathcal{C}_v$ resp. $\mathcal{C}_a$ the set of central vertices resp. central arcs in Γ (cf. def. before thm. 2.3).

Corollary 4.3. *From thm. 2.3 we know that the components of the strict transform $C_1(D)'$ of $C_1(D)$ on the resolution X of $(S,0)$ go through components corresponding to elements of $\Gamma_{NT} \cup \mathcal{C}_a \cup \mathcal{C}_v$, and we also know the number of branches of $C_1(D)'$ on each of these components.*

From thm. 4.1 we know the contact between the corresponding branches of $C_1(D)$ (or $\Delta_{S,0}$) : the contact between two branches whose strict transforms lie respectively on a component L_x and a component L_y equals $1 + N$, where N is the number of blow-ups necessary so that L_x and L_y no longer be in the same Tyurina component of the corresponding $(S_N, 0_N)$ singularity, with the further requirement that if ,say, the first branch actually goes through a central arc $L_x \cap L_{x'}$, the number N corresponds to the number of blow-ups so that both *x and x' no longer be in the same Tyurina component as y.*

From this, we easily see that each $x_i \in \Gamma_{NT}$ contributes with a $\delta_{x_i} := \delta_{w(x_i)-\gamma(x_i)}$ (cf. 4.2 ii)), i.e. $2(w(x_i) - \gamma(x_i)) - 2$ lines, and that the contact between these δ_{x_i} and other branches of the discriminant is one. For the contribution of the central elements, we first compute the number of branches on each components with thm. 2.3 and use thm. 4.1 for the contact as in the following examples :

Example 1. Consider $(S,0)$ with the graph Γ as on figure 1 below, where the $\bullet$ denote Tyurina vertices (with $w(x) = \gamma(x)$), and $\Gamma_{NT} = \{x_1, \ldots, x_4\}$ with the weights indicated on the graph. Remark that as a general rule $\delta_{x_i} = \emptyset$ when $w(x_i) = \gamma(x_i) + 1$, hence here only x_1 actually gives a δ_{x_1}equals to four lines.
i) In the *first Tyurina component* (bounded by x_1, x_2, x_4) there is a central vertex and a central arc, which respectively give a Δ_{A_5} and a Δ_{A_4} curve.

After two blow-ups the boundaries of the central arc and the central vertex are in distinct Tyurina components, hence the contact between the Δ_{A_5} and Δ_{A_4} is three.
ii) In the *second Tyurina component* (bounded by x_2 x_3), there is a central vertex : this gives a Δ_{A_3} which has contact 1 with the others Δ_{A_i} obtained.

Hence, using coordinates, we may give as representative of the equisingularity class of $\Delta_{S,0}$:

$\Delta_{S,0}$: $(x^4 + y^4)(x^2 + y^6)(x^2 + y^5)(y^2 + x^4) = 0.$

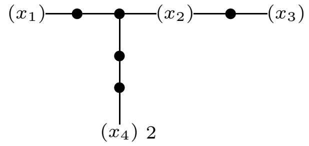

Figure 1

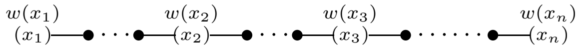

Figure 2

Example 2. If $(S,0)$ is a cyclic-quotient singularity i.e. has a graph Γ as on figure 2, we may order $\Gamma_{NT} = \{x_1 < x_2 < \cdots < x_n\}$ and each central element x (central vertex or central arc) lies in a unique $[x_i, x_{i+1}]$ and is easily seen to contribute to $\Delta_{S,0}$ by a $\Delta_x := \Delta_{A_{l([x_i,x_{i+1}])}}$, where $l[x_i, x_{i+1}]$ is the number of vertices on the chain $[x_i, x_{i+1}]$; the contact between each Δ_x is one (i.e their tangent cones have no common components). Here δ_{x_i} is $2w(x_i) - 4$ lines for $i = 1$ and $i = n$, and $2w(x_i) - 6$ for $1 < i < n$, all this lines being distinct. So, with Δ_{A_n} as in notation 4.2 i) :

$$\Delta_{S,0} = \delta_{x_1} \cup \Delta_{A_{l[x_1,x_2]}} \cup \delta_{x_2} \cup \cdots \Delta_{A_{l[x_{n-1},x_n]}} \cup \delta_{x_n},$$

with contact one between all the curves in the "$\cup$".

Remark 1. In particular, the equisingularity type of $(\Delta_{S,0}, 0)$ depends only on the resolution graph of $(S,0)$ i.e. of the topological type of $(S,0)$, what is known to be wrong for other normal surface singularities as shown in [6].

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